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The Secretary Problem and its Extensions: A Review

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Summary

The development of what has come to be known as the secretary problem is traced from its origins in the early 1960's. All published work to date on the problem and its extensions is reviewed.

Key words: Candidate problem; Dowry problem; Dynamic programming; Googol; Hotel problem; Marriage problem; Optimal stopping; Secretary problem.

1 The standard problem

1.1 Introduction

What I shall call the standard secretary problem is as follows. A known number of items is to be presented one by one in random order, all $n!$ possible orders being equally likely. The observer is able at any time to rank the items that have so far been presented in order of desirability. As each item is presented he must either accept it, in which case the process stops, or reject it, when the next item in the sequence is presented and the observer faces the same choice as before. If the last item is presented it must be accepted. The observer's aim is to maximize the probability that the item he chooses is, in fact, the best of the n items available. We shall abbreviate this outcome to the single word 'win'.

Since the observer is never able to go back and choose a previously-presented item which, in retrospect, turns out to be best, he clearly has to balance the danger of stopping too soon and accepting an apparently-desirable item when an even better one might be still to come, against that of going on too long and finding that the best item was rejected earlier on. The obvious application to choosing the best applicant for a job gives the problem its common name, although many other names, such as marriage, dowry, beauty contest and candidate, have been used to describe the equivalent problem in other contexts.

We defer to feminine sensitivities by referring throughout to 'items' rather than 'secretaries' but for the sake of definiteness we use the masculine pronoun for the observer.

1.2 Historical note

The origins of the secretary problem are obscure. Gilbert & Mosteller (1966) relate that A. Gleason posed the problem in 1955, having himself heard it from somebody else. Gardner (1960a, b) attributes the problem to J.H. Fox and L.G. Marnie in 1958 and gives a compressed account of a complete analysis of the problem by L. Moser and J.R. Pounder. Bissinger & Siegel (1963) posed the special case with $n = 1000$ and the solution

was given by Bosch (1964) and by 12 other people. The first published solution of the standard problem was given by Lindley (1961).

A remarkable forerunner of this modern work was a problem posed by Cayley (1875) in which a sequence of values drawn independently from a known probability distribution is presented to the observer. It was solved for the uniform distribution by Moser (1956) and for general distributions by Guttman (1960). We shall return to this so-called 'full information' problem later.

1.3 Solution of the standard problem

The state of the process at any time may be described by two numbers (r, s), where r is the number of items so far presented and s the apparent rank of the rth, last presented item. If $s \neq 1$ there is obviously no point in accepting the rth item as it cannot possibly be the best. After the next item has been presented the new state of the process will be $(r+1, s')$, where s' is equally likely to be any one of the values $1, 2, \ldots, r+1$. If $s = 1$, this item is a *candidate* for acceptance. The probability that it is in fact the best of all n items is just r/n. Letting $V(r, s)$ denote the maximum expected probability of choosing the best item when the state of the process is (r, s), the principle of dynamic programming yields the equations

$$V(r, 1) = \max \left\{ \frac{r}{n}, \frac{1}{r+1} \sum_{s'=1}^{r+1} V(r+1, s') \right\}, \tag{1.1}$$

$$V(r, s) = \frac{1}{r+1} \sum_{s'=1}^{r+1} V(r+1, s') \quad (s = 2, 3, \ldots, r), \tag{1.2}$$

with $V(n, s) = 1$ if $s = 1$ and 0 otherwise.

Lindley (1961) solved these by simple backward recursion over $r = n, n-1, \ldots, 1$. If we define

$$a_r = \frac{1}{r} + \frac{1}{r+1} + \ldots + \frac{1}{n-1}, \tag{1.3}$$

the optimal action in state $(r, 1)$ is to stop if $a_r < 1$ and to continue if $a_r > 1$. Thus, if r^* is the integer r for which $a_{r-1} \geqslant 1 > a_r$, the optimal policy is to reject the first $r^* - 1$ items and then to accept the first item thereafter that is better than all previous items.

The probability of winning using this policy is $(r^* - 1)a_{r^*-1}/n$. As $n \to \infty$, both this and $r^*/n \to e^{-1} = 0.368. \ldots$

Gilbert & Mosteller (1966) show that $\tilde{r} = [(n - \frac{1}{2})e^{-1} + \frac{1}{2}]$ is a better approximation to r^* than $[ne^{-1}]$ although the difference is never more than 1.

1.4 Alternative solution

Although most succeeding papers employ direct algebraic methods similar to Lindley's, a completely different approach due to Dynkin (1963) provides an alternative tool. This uses the fact that the stages $r(0) = 1, r(1), r(2), \ldots$ at which candidates are observed form a Markov chain, since

$$p(r(i+1) = l \mid r(0) = 1, r(1) = a, \ldots, r(i) = k)$$

is just the probability that the $(k+1)$th, $(k+2)$th, $\ldots, (l-1)$th items are less desirable than the kth and the lth item is more desirable, and so does not depend on $r(0), \ldots, r(i-1)$.

In fact

$$p_{kl} = p(r(i+1) = l \mid r(i) = k) = \frac{p(r(i) = k \text{ and } r(i+1) = l)}{p(r(i) = k)}$$

$$= \frac{1/l(l-1)}{1/k} \quad (1 \leqslant k < l \leqslant n)$$

since the numerator is the probability that the kth and lth items are the second-best and best of the first l items. At $r(i) = k$, the probability of winning if observation stops is k/n, while if observation continues until $r(i+1)$ and then stops the probability is

$$\sum_{l=k+1}^{n} p_{kl} \frac{l}{n} = \frac{k}{n} a_k.$$

The one-step-ahead rule compares these probabilities and is, in fact, optimal since the conditions for the monotone case of Chow, Robbins & Siegmund (1971) are satisfied. Note, however, that it is necessary to look ahead from one relatively best item to the next, since the one-*item*-ahead policy is clearly suboptimal.

1.5 Rank of accepted item

Bartoszynski (1974) elegantly obtains a combinatorial identity by calculating the maximal probability of winning in two different ways, and then shows that if one of the first $n-1$ items is selected by the optimal policy, the probability that its true rank is t is given by

$$p(t) = \binom{n}{r^*-1}^{-1} \sum_{i=t+1}^{n-r^*+2} \left\{ \binom{n-i}{r^*-2} \Big/ (i-1) \right\} \quad (t = 1, \ldots, n-r^*+1),$$

which decreases as t increases.

2 Simple extensions

A few extensions to the standard problem turn out to be quite easy to solve. The basic characteristics of finite n, a single choice and the goal of maximizing the probability of winning remain unchanged.

2.1 Uncertain employment and recall

Smith (1975) introduced the possibility that any item, if accepted, has some probability of not being available, in which case it has to be passed over and the next item observed.

Yang (1974) allowed the observer at any stage to go back and try to accept an item which had been previously rejected. If it is available it is accepted but otherwise it remains unavailable ever after and the observer must continue inspecting new items.

These two possibilities were allowed simultaneously by Petrucelli (1981).

The state of the process is again described by (r, s), but s now denotes the number of items before the rth at which the best item so far was seen. If that best item has already been found to be unavailable, s is set to ∞. It is clearly only necessary to consider soliciting the best item so far. If this is done, the probability that it is available is $q(s)$, where $q(\infty) = 0$ and $q(0) \leqslant 1$.

The dynamic programming equation is now

$$V(r, s) = \max \left[\frac{r}{n} q(s) + V(r, \infty)(1 - q(s)), \frac{1}{r+1} V(r+1, 0) + \frac{r}{r+1} V(r+1, s+1) \right], \quad (2.1)$$

where the first term corresponds to trying to accept the best item so far and the second to observing the next item. Boundary conditions are

$$V(n, s) = q(s), \quad V(r, \infty) = \frac{1}{r+1} V(r+1, 0) + \frac{r}{r+1} V(r+1, \infty).$$

Petrucelli proves three general properties of the optimal policy and gives explicit solutions for two important special cases:

Case 1: $q(0) = q$, $q(s) = p$ for $s = 1, 2, \ldots, n$, where $0 < p < q$. Here the optimal policy takes the form 'reject the first $r^* - 1$ items; try to accept all items with apparent rank 1 observed thereafter; if all n items have been observed and the actual best was in the first $r^* - 1$ items, go back and try to accept it', where r^* is the smallest integer r such that

$$\prod_{k=r}^{n-1} \left(1 + \frac{1-q}{k}\right) \leq \frac{q - p(1-q)}{q^2}.$$

As $n \to \infty$, $r^*/n \to V$, where

$$V = \left[\frac{q^2}{q - p(1-q)}\right]^{(1-q)^{-1}},$$

and the maximum probability of winning tends to $V\{q - p(1-q)\}/q$.

Case 2: $q(r) = qp^r$. Here items become increasingly likely to be unavailable the further back in the sequence they lie, so the observer cannot afford the luxury of waiting until he has seen all n items before making his recall unless p is very close to 1. The form of the optimal policy is now as follows.

'If $q/(1-p) < n - 1$, observe the first r^* items, then try to accept the best of them; if this is not available, continue and try to accept all items with apparent rank 1 observed thereafter. If $q/(1-p) > n - 1$ observe all n items and then try to accept the best.'

Explicit formulae are again given for r^* and for the probability of winning. These both tend to $q^{(1-q)^{-1}}$ as $n \to \infty$, the same value as in special case 1 above with $p = 0$, so asymptotically the recall facility confers no advantage.

These two examples are explicitly soluble because the optimal policy involves use of recall at most once.

A similar version of recall was considered by Smith & Deely (1975) in which any one of the last m items observed can be accepted (they are certain to be available). The process must, however, stop when the observer chooses this option and this has the effect of deleting the expression $V(r, \infty)(1 - q(s))$ from (2.1). The observer clearly need only consider stopping when $s = m - 1$, that is when the best item so far is about to become unavailable. As with previous policies, the first r^* items should be rejected. It is not possible to get a closed expression for this, but an algorithm for finding it and the maximum probability of winning is given. If $m/n = \alpha \geq \frac{1}{2}$ then $r^* = m$ and the probability of winning is

$$V(1, 1) = 2 - \frac{m}{n} - a_m,$$

which tends to $2 - \alpha - \log \alpha$ as $n \to \infty$, while if m is fixed then r^*/n and $V(1, 1)$ both tend to e^{-1}, giving no asymptotic advantage over the standard no recall problem.

2.2 Discounting

Rasmussen & Pliska (1976) introduce a discount factor d, so the expected 'gain' from stopping in state $(r, 1)$ is now $d^r(r/n)$. With this modification to (1.1) the analysis goes

through giving an optimal policy of the same form, but with r^* now the smallest integer r such that

$$\sum_{k=r}^{n-1} d^{k+1-r}/k \leq 1.$$

As $n \to \infty$, r^* increases to a finite upper limit $r^{**}(d) < (1-d)^{-1}$ and, as $d \to 1$, $r^{**}(d)$ itself increases to ∞. It can be closely approximated by $-0.4348/\log d$ when d is close to 1. The optimal expected gain now tends to 0 as $n \to \infty$, in marked contrast to that of the standard problem. The discount factor forces the observer to stop earlier than he otherwise would and when n is large this gives him very little chance of getting the best item.

3 Minimizing the expected rank of the accepted item

The objection to the goal of maximizing the probability of accepting the best item is that this implies a utility function that takes the value 1 if the best item is accepted and 0 otherwise. Lindley aptly names this the goal of 'nothing but the best'. A more realistic utility function is the one that takes the value $n - i$ when the ith best item is accepted. Maximizing expected utility then corresponds to minimizing expected rank of the accepted item.

If the observer chooses to stop in state (r, s) and accept the item that is sth best out of the first r items, the probability that its true rank out of all n items is i is

$$p_{i,n}(r, s) = \binom{i-1}{s-1}\binom{n-i}{r-s} \bigg/ \binom{n}{r} \quad (i = s, s+1, \ldots, n+s-r), \tag{3.1}$$

so that the expected utility is

$$U(r, s) = \sum_{i=s}^{n+s-r} (n-i)p_{i,n}(r, s) = n - \frac{n+1}{r+1} s. \tag{3.2}$$

The dynamic programming equations corresponding to (1.1) and (1.2) are

$$V(r, s) = \max\left\{U(r, s), \frac{1}{r+1}\sum_{s'=1}^{r+1} V(r+1, s')\right\}, \quad V(n, s) = n - s. \tag{3.3}$$

Since the first term is a decreasing function of s while the second is independent of s, the optimal policy is of the form 'for each r, stop and accept the latest item if its apparent rank $s \leq s^*(r)$, continue if $s > s^*(r)$'.

The solution is due to Lindley (1961) who gave a recurrence equation by which $s^*(r)$ may be calculated. He tried approximating this by a differential equation in $x = r/n$ as $n \to \infty$, but was unable to make this accurate enough to yield useful results.

It was Chow *et al.* (1964) who first showed that, as $n \to \infty$,

$$V_n = V(0, 0) = V(1, 1) \to \prod_{j=1}^{\infty} \left(\frac{j+2}{j}\right)^{1/(j+1)} = 3.8695$$

and that V_n is in fact an increasing function of n.

They obtained these results by direct methods since, although they too had a heuristic argument involving approximation of difference by differential equations, they had not been able to make it rigorous.

4   General utility function

The two preceding sections are simple special cases of the problem in which the observer received U_i units of utility (or payoff) if the accepted item is the ith best. It seems common-sense to assume that U_i is nonincreasing in i, and the form of the optimal policy in this case was first given by Mucci (1973a). If the observer accepts the rth item whose apparent rank is s, his expected utility is

$$Q(r, s) = \sum_{i=s}^{n+s-r} U_i p_{i,n}(r, s).$$

Since this is decreasing in s, substituting it for $U(r, s)$ in (3.3) gives the same kind of optimal policy as before. Moreover, $Q(r, s)$ is decreasing in r so that the critical numbers $s^*(r)$ must increase with r. Another way of describing the optimal policy is therefore to define an increasing sequence of numbers $r_1 \leqslant r_2 \leqslant \ldots \leqslant r_n$ such that $s^*(r) = i$ if and only if $r_i < r \leqslant r_{i+1}$. One should, therefore, stop when in state (r, s) only if $r > r_s$. The first r_1 items are thus rejected, the relatively best is accepted if one such appears between the $(r_1 + 1)$th and the r_2th items, the relatively best or second-best between the $(r_2 + 1)$th and the r_3th items, and so on.

This optimal policy had been found for the special case

$$U_1 = U_2 = \ldots = U_k = 1, \quad U_{k+1} = \ldots = U_n = 0$$

(that is, choice of any one of the k best items is called a win) by Gusein-Zade (1966). He showed that for $k = 2$ the optimal expected utility tends to $2\phi - \phi^2 = 0.574$ as $n \to \infty$ where $\phi = 0.347$ is the root of the equation $\phi - \log \phi = 1 - \log \frac{2}{3}$. Also $s^*(r) < 1$ while $r/n < \phi$, and $s^*(r) = 1$ while $\phi < r/n < \frac{2}{3}$, so $r_1/n \to \phi$ and $r_2/n \to \frac{2}{3}$ as $n \to \infty$. These same results for $k = 2$ were given independently by Gilbert & Mosteller (1966). Bartoszynski (1976) again obtains a combinatorial identity by calculating the probability of winning in two different ways when $k = 2$.

For general k, Gusein-Zade showed that the limiting (as $n \to \infty$) optimal expected utility tends to 1 as $k \to \infty$ at least as fast as $1 - k^{-1} \log k$. Frank & Samuels (1980) computed the utilities for $k = 1$ to 25 and these strongly suggested exponentially fast convergence. Note that these had previously been computed for k up to 10 by Rasmussen (1972). They were indeed able to prove this, showing that for any k the optimal expected utility is $1 - [1 - t_1(k)]^k$, where $t_1(k)$ is the limiting value of r_1/n as $n \to \infty$ for that value of k. Even more surprisingly, they showed that for any fixed j, as $k \to \infty$,

$$\lim_{n \to \infty} (r_j - r_1)/n \to 0,$$

so that for large k the limiting optimal policy as $n \to \infty$ stops very soon after r_1 items have been observed. Two open problems remain. Does $t_1(k)$ decrease monotonically to its limit as $k \to \infty$? Is $t_j(k) = \lim r_j/n$ as $n \to \infty$ monotonic decreasing with k?

The other achievement of Mucci (1973a) was finally to rigorously derive the limiting differential equation form of the dynamic programming equation. Starting from

$$V(r, s) = \max \left\{ Q(r, s), \frac{1}{r+1} \sum_{s'=1}^{r+1} V(r+1, s') \right\} \quad V(n, s) = U_s,$$

and writing

$$f_n\left(\frac{r}{n}\right) = \frac{1}{r} \sum_{s'=1}^{r} V(r, s')$$

we have

$$f_n\left(\frac{r}{n}\right) - f_n\left(\frac{r+1}{n}\right) = \frac{1}{r} \sum_{s'=1}^{r} \left\{Q(r, s') - f_n\left(\frac{r+1}{n}\right)\right\}^+,$$

where $x^+ = \max(x, 0)$. Letting $n \to \infty$ we obtain the limit

$$f'(x) = -\frac{1}{x} \sum_{s'=1}^{\infty} \{R_{s'}(x) - f(x)\}^+, \quad f(1) = 0, \tag{4.1}$$

where

$$R_s(x) = \sum_{l=s}^{\infty} U_l \binom{l-1}{s-1} x^s (1-x)^{l-s}. \tag{4.2}$$

The optimal expected utility $V(0, 0) = f_n(0)$ satisfies

$$|V(0, 0) - f(0)| \leq 10 n^{-\frac{1}{2}} \log n + 30 U_{[c \log n]},$$

where c is a given constant, and the numbers $r_1 \leq r_2 \leq \ldots \leq r_n$ that determine the optimal policy are indicated, for large n, by the limits

$$x_i = \lim_{n \to \infty} r_i/n,$$

where the x_i are uniquely determined by $R_i(x_i) = f(x_i)$.

In a further paper, Mucci (1973b) allowed the utilities to form an unbounded decreasing sequence, corresponding to acceptance of poor items being positively harmful. He showed that so long as the utilities decrease no faster than a polynomial of finite order, the optimal expected utility remains finite as $n \to \infty$.

5   Unknown number of items

If n is unknown, the observer faces an additional risk. If he rejects any item, he may then discover it was the last one, in which case he receives nothing at all.

5.1   Known prior distribution

Letting N denote the unknown true number of items, it is assumed that $p_i = p(N = i)$ $(i = 1, 2, \ldots)$ are known to the observer. Write

$$\pi_k = p(N \geq k) = \sum_{i=k}^{\infty} p_i.$$

Presman & Sonin (1972) provided a treatment of the standard problem, using the Dynkin approach. The transition probabilities of the imbedded Markov chain are

$$p(k, l) = \frac{k \pi_l}{l(l-1) \pi_k} \quad (k < l < \infty), \quad p(k, \infty) = \sum_{i=k}^{\infty} \frac{k}{i} \frac{p_i}{\pi_k},$$

where the absorbing state '∞' has to be introduced to cover the possibility that $k \leq N < l$. In this case, the kth item is the actual best, and the probability of this, given that it is relatively best, is just $p(k, \infty)$.

The one-step-ahead policy is no longer optimal, and the optimal policy is no longer simple. The set Γ of states k at which it is optimal to stop may be thought of, trivially, as a succession of 'islands' separated by a sea of continuation states. The key to the form of Γ

is held by the numbers

$$c_k = \sum_{i=k}^{\infty} \left(p_i - \sum_{j=i+1}^{\infty} \frac{p_j}{j} \right) \Big/ i = \sum_{i=k}^{\infty} d_i/i, \tag{5.1}$$

say. If there is some k^* such that $c_k \geq 0$ for all $k \geq k^*$, then all states from k^* to ∞ belong to Γ and the optimal policy consists of a finite number of islands only. Moreover if the $\{d_i\}$ sequence changes sign from $-$ to $+$ N_0 times, then Γ has no more than N_0 islands. For three special cases, where N is uniform, geometric or Poisson, the d's change sign exactly once so the optimal policy returns to its standard form. For the uniform distribution from 1 to n, the single cutoff value $k^* \simeq ne^{-2}$ and $p(\text{win}) \rightarrow 2e^{-2} = 0.2707\ldots$ as $n \rightarrow \infty$.

Gianini-Pettitt (1979) considered the expected rank problem. She showed that the optimal policy is still as in § 3 above, but $\{s^*(r)\}$ need no longer be an increasing sequence. When n is known, the minimum expected rank is an increasing function of n, but for two variables N and N' with N stochastically smaller than N' it does not necessarily follow that the minimum expected rank for the N problem is less than that for N'.

For the particular family of priors

$$p(N = i \mid N \geq i) = (n - i + 1)^{-\alpha} \quad (i = 1, 2, \ldots, n)$$

in which $\alpha = 1$ gives the uniform distribution, it was shown that the limiting minimum expected rank is ∞ if $\alpha < 2$ and the Chow et al. (1971) limit 3.8695 if $\alpha > 2$ so that not knowing N is asymptotically no disadvantage. When $\alpha = 2$ the lim inf is some number greater than 3.8695. Within this family it remains an open question whether the minimum expected rank is an increasing function of n.

Two papers in this area, Rasmussen (1975) and Rasmussen & Robbins (1975) are wrong, and presumably published in ignorance of Presman & Sonin (1972). Irle (1980) gives a counter-example and goes on to introduce a third basic approach to solving the secretary problem, derived by Rasch (1975) from Howard's policy iteration method. If we use a discount factor d as in § 2.2, numbers similar to (5.1)

$$c_k(d) = \sum_{i=k}^{\infty} p_i \left(d^k - \sum_{l=k}^{i-1} \frac{d^{l+1}}{l} \right) \Big/ i$$

are defined and conditions on them are given that cause the iteration to converge at the 2nd, 3rd or 4th cycle, together with the corresponding optimal policies.

5.2 Admissible policies

Abdel-Hamid, Bather & Trustrum (1982) are concerned with the relation between Bayes policies as found by Presman & Sonin and those a non-Bayesian might consider by treating N as an unknown parameter, thereby pursuing the close analogy with standard ideas in decision theory. They define a randomized policy Π that accepts the rth item, if it is a candidate, with probability q_r. The probability that all of the first r items will be rejected is therefore

$$U_r(\Pi) = (1 - q_1)(1 - \tfrac{1}{2}q_2) \ldots \left(1 - \frac{q_r}{r} \right),$$

and if $N = n$ the probability of winning with this policy is

$$V_n(\Pi) = \frac{1}{n} \sum_{r=1}^{n} q_r U_{r-1}(\Pi).$$

The policy Π is then *admissible* if there exists no other policy Π' such that $V_n(\Pi') \geqslant V_n(\Pi)$ for all $n \geqslant 1$ with strict inequality for at least one n.

The paper's key result is that Π is admissible if and only if $U_r(\Pi) \rightarrow 0$ as $r \rightarrow \infty$. In terms of the q's an equivalent condition is that either $q_1 = 1$ or $q_1 < 1$ and $\Sigma\, q_r/r$ diverges.

If now N has a prior distribution as above, generically denoted by p, the expected probability of winning with policy Π is

$$A(p, \Pi) = \sum_{n=1}^{\infty} p_n V_n(\Pi)$$

and the Bayes reward is

$$B(p) = \sup_{\Pi} A(p, \Pi).$$

By taking the proper prior

$$p_m(N = n) = \begin{cases} c/(n+1) & (1 \leqslant n \leqslant m-1), \\ c & (n = m), \\ 0 & (n > m), \end{cases}$$

with $c = (\sum_i i^{-1})^{-1}$, where the sum is over $i = 1, \ldots, m$, it is easy to show that

$$\inf_p B(p) = 0,$$

but by adding the single improper prior obtained by fixing $c > 0$ and letting $m \rightarrow \infty$ to the class of all proper priors, the extended Bayes policies now constitute the whole family of admissible policies. This follows by first establishing the fact that Π is an extended Bayes policy if and only if $U_r(\Pi) \rightarrow 0$.

5.3 Random arrivals

Another way in which the number of items may be unknown is if they are presented at the time points of a Poisson process of known rate λ and the observer must make his choice before some fixed time T. His decision on any item will now be heavily influenced by how much time t he has left for choosing later items.

Karlin (1962) and Sakaguchi (1976) first considered such problems, but only treated the 'full-information' case, see §9. Cowan & Zabczyk (1978), using the Dynkin approach, define a homogeneous Markov process $\{X_n\}$ with $X_n = (r, t)$ meaning that the rth item is observed at time $T - t$ and is the nth candidate. The one-step-ahead policy is again optimal and tells the observer to stop at the first candidate for which $\lambda t \leqslant x(r)$, where $x(r)$ is the unique solution of

$$\sum_{n=0}^{\infty} \frac{x^n}{n!\,(r+n)} = \sum_{n=1}^{\infty} \frac{x^n}{n!\,(r+n)} \sum_{k=1}^{n} \frac{1}{k+r-1}.$$

A table of these values is given for r up to 45.

Stewart (1981) takes a different formulation in which an unknown number of N items arrive at times which are all independently exponentially distributed with known mean $1/\lambda$. Given a prior distribution $p_0(n) = p(N = n)$, the arrival times $t_1, \ldots, t_r$ of the first r items provide information about N and lead to a 'current distribution'

$$p_r(n \mid t_1, \ldots, t_r) = p(N = n \mid T_1 = t_1, \ldots, T_r = t_r)$$

via Bayes theorem.

Taking the prior to be uniform over $n = 0, 1, \ldots, M$ and then letting $M \to \infty$ yields the posterior

$$p_r(n \mid t_1 \ldots t_r) = \begin{cases} \binom{n}{r}\{1 - \exp(-\lambda t_r)\}^{r+1} \exp\{-(n-r)\lambda t_r\} & (n \geqslant r), \\ 0 & (n < r), \end{cases}$$

depending only on t_r. The state of the process has to be (r, s, t_r) and the dynamic programming equation becomes

$$V(r, s, t_r) = \max\left[E_{N|r,t_r}(r/n), E_{S_{r+1}, T_{r+1}|r,s,t_r}\{V(r+1, s_{r+1}, t_{r+1})\}\right]$$

since the rank s_{r+1} of the $(r+1)$th item, and the time t_{r+1} at which it will be presented, now depend on N. They do not, however, depend on the ordering of the first r items so the second term is independent of s. The first term is $1 - e^{-\lambda t_r}$ if $s_r = 1$ and 0 otherwise, from which it follows that the optimal policy is to accept the first item for which $s_r = 1$ and $t_r > \tau^* = 0.4587/\lambda$. This has close affinities with the standard problem, in that given $N = n$, the expected number of items arriving before time τ^*, and hence automatically rejected, is n/e and the probability of winning again tends to $1/e$ as $n \to \infty$. If λ is unknown, then using any value $\hat\lambda$ within a factor of 2 of the true value still keeps the probability of winning larger than 0.3.

6   More than one choice

Once the observer is allowed to accept more than one item, many possible problems suggest themselves. Gilbert & Mosteller (1966) first solved some of them and their cornucopia of interesting results stimulated much further work.

6.1   k choices, win if any of them is the best

Sakaguchi (1978) gave a simpler derivation of Gilbert & Mosteller's results by using Dynkin's approach and showing the one-step-ahead policy is optimal. We define state (r, s) to mean the rth item is observed, is a candidate and the observer still has s choices to make. Absorbing states (∞, s) meaning the nth item has been observed and is not a candidate, and $(\infty, 0)$ meaning all k choices have been made, have to be added. Transition probabilities from state (r, s) if the observer accepts are $r/j(j-1)$ to state $(j, s-1)$ and r/n to state $(\infty, s-1)$, while if the observer rejects the same probabilities lead to state (j, s) and (∞, s) respectively. The dynamic programming equation is

$$V(r, s) = \max\left[\frac{r}{n} + \sum_{j=r+1}^{n} \frac{r}{j(j-1)} V(j, s-1), \sum_{j=r+1}^{n} \frac{r}{j(j-1)} V(j, s)\right],$$

and the one step ahead policy can be evaluated by considering $s = 1, 2, \ldots$ successively. These determine a set of numbers $r_k^* \leqslant r_{k-1}^* \leqslant \ldots \leqslant r_2^* \leqslant r_1^*$ such that the observer makes his $(k - s + 1)$th choice at the first candidate to appear after item $r_s^* - 1$. The value of r_1^* is, of course, that for the standard problem, while $r_2^*/n \to e^{-2/3} = 0.2231$. Gilbert & Mosteller give numerical results up to $k = 8$ and show that the much simpler policy of accepting the first k candidates to appear after item $r^* - 1$, with r^* chosen optimally, is very nearly as good as the optimal policy.

6.2   k choices, minimize sum of actual ranks

Henke (1970) showed that the rth item should be accepted, when j items have already been accepted, only if its relative rank $s < s_{r,j}^*$ and gave a system of recurrence equations that determine these critical values.

6.3 Two choices, win if they are best and second best

This problem was solved, apparently independently, by Nikolaev (1977) and more explicitly by Tamaki (1979a). The optimal policy says choose the first two candidates to appear after the first $r_1^* - 1$ items or, as second choice, the first item with apparent rank 2 to appear after the first $r_2^* - 1$ items. Formulae are given for r_1^*, r_2^* and the probability of winning. As $n \to \infty$, $r_1^*/n \to 0.2291$, $r_2^*/n \to e^{-1/3} = 0.6065$, $p(\text{win}) \to 0.2254$.

Sakaguchi (1979) generalizes slightly by supposing that each item has probability q of being available if chosen, as in § 2.1. The form of the optimal policy remains unchanged, but now $r_1^*/n \to \theta$, the unique root of a rather complicated equation in q, and $r_2^*/n \to \phi = q^{1/\{2(1-q)\}}$ as $n \to \infty$. The probability of winning tends to

$$\frac{\theta}{2-q} (2\phi^{2-q} - q\theta^{2-q}),$$

a smaller value than Smith (1975) found for the equivalent one-choice problem.

6.4 Two choices, win if either is best or second best

Tamaki (1979b) solved this rather more favourable problem using the usual recursive dynamic programming approach. The optimal policy is again sensible, in that with two choices to make the observer should accept the rth item provided $r \geqslant r_1^*$ if its relative rank $s = 1$ and $r \geqslant r_2^*$ if $s = 2$, while with one choice left to make, the initial values are some other numbers r_1^{**}, r_2^{**}. The latter values were again found earlier by Gilbert & Mosteller. Limiting values as $n \to \infty$ are given, and the probability of winning tends to 0.7934.

7 The infinite problem

We have already described many asymptotic results as $n \to \infty$. In a sense, then, we have the 'infinite solution' as the limit of finite solutions, but have not yet mentioned the infinite problem for which this is the optimal policy. Gianini & Samuels (1976) provided the answer.

7.1 Limits of finite problem

The statement 'n items are presented in random order' does not immediately suggest a limiting problem, but the equivalent statement 'each item is equally likely to be presented first, second, third, etc.' is more helpful. We therefore let z_i denote the time at which the ith best item is presented, and suppose that $\{z_i\}$, for $i = 1, 2, \ldots$, are independently uniformly distributed in the interval $[0, 1]$. Accepting an item of true rank i yields utility U_i, with $\{U_i\}$ a decreasing sequence. The maximum expected utility $f(t)$ of the optimal policy from time t onwards satisfies Mucci's differential equation (4.1) so that, when

$$V = \lim_{t \downarrow 0} f(t)$$

is finite, the optimal policy chooses times $0 < t_1 \leqslant t_2 \leqslant \ldots \leqslant 1$ and accepts the first item presented in the time interval $[t_s, t_{s+1})$ that has relative rank of s or better. If $u = \lim U_i$ as $i \to \infty$ is finite then V is finite and $f(t)$ is the unique bounded solution of the left-hand equation (4.1) with the boundary condition $f(1) = u$. If the $\{U_i\}$ decrease like a power of i then V is finite and $f(t)$ is finite for all $t < 1$ but tends to ∞ as $t \uparrow 1$.

Lorenzen (1978) generalizes the infinite problem by allowing the expected utility of stopping at time t and accepting an item of relative rank s to be any function $A_s(t)$, rather

than the particular function $R_s(t)$ of (4.2). The only restrictions are: $A_1(0) = A_2(0) = \ldots = A_\infty(0+)$, $A_i(t) \geqslant A_{i+1}(t)$ for all i, and $A_i(t)$ is continuous and finite on some interval $(b_i, 1)$. An example might be where $h(t)$ denotes a cost of observing items up to time t, so that $A_i(t) = R_i(t) - h(t)$.

The same differential equation holds and the optimal rule can again be stated in the form 'stop at time t if the item then presented has relative rank s satisfying $A_s(t) \geqslant f(t)$'. This no longer gives a simple cut-off rule, but an island rule analogous to § 5.1, since $A_s(t)$ is now not necessarily decreasing in t. For example, with $U_1 = 1$, $U_2 = U_3 \ldots = 0$ and sampling cost

$$h(t) = \begin{cases} 0 & (0 \leqslant t \leqslant \frac{1}{3}), \\ 1 - e^{-1} & (\frac{1}{3} \leqslant t \leqslant 1), \end{cases}$$

the optimal policy accepts the first candidate to appear in either of the intervals $[\frac{1}{3}e^{-1}, \frac{1}{3})$ or $[1/e, 1]$ and to reject all items at other times.

Lorenzen was able to show that the maximum expected utility for the finite problem tends to that for the infinite problem provided $\{A_s(.)\}$ is an equicontinuous family, but whether it does so for completely general $A_s(.)$ remains unknown.

7.2 Partial recall with full or finite memory

Gianini (1977) introduces the following interesting discretization of the infinite problem that allows the observer a certain amount of recall of past items. It also neatly imbeds the finite problem in the infinite one.

Suppose the interval $[0, 1]$ is divided into n equal subintervals

$$\left(\frac{k-1}{n}, \frac{k}{n}\right] \quad (k = 1, 2, \ldots, n).$$

At the end of each subinterval the observer must choose either to stop and accept the best item presented in that subinterval or to continue until the end of the next subinterval. This choice may be based on the full memory of the relative ranks of all items presented by that time. It again happens that the optimal policy is of the cut-off form 'At time k/n stop if the relative rank s of the best item in the previous subinterval is less than or equal to s_k'. The value s_k is the smallest s such that $R_s(k/n) \geqslant v(n, k)$, where $R_s(.)$ is given by (4.2) and $v(n, k)$ is defined by recurrence relations

$$v(n, k-1) = \sum_{j=1}^{\infty} \frac{1}{k} \left(\frac{k-1}{k}\right)^{j-1} \max\left\{R_j\left(\frac{k}{n}\right), v(n, k)\right\}, \qquad v(n, n-1) = \sum_{j=1}^{\infty} \frac{1}{n} \left(\frac{n-1}{n}\right)^{j-1} U_j.$$

The maximum expected utility of this policy $v(n, 0) \geqslant V$ for the infinite problem.

Now suppose the observer is further constrained in that his decision to stop or continue can be based not on all the items he has seen but only on the relatively best items in each of the preceding subintervals. Let T_k be the time at which the relatively best item in the subinterval $((k-1)/n, k/n]$ is presented, $Y_1(k)$ the relative rank of that item among the first k relatively best items, i.e. those presented at times $T_1, \ldots, T_k$, $X_1(k)$ the absolute rank of that item among the n relatively best items, and $Y_2(k), X_2(k)$ the relative and absolute ranks of that item among all items, not just the relatively best ones. We seek a stopping rule τ to maximize the expected value of $U_{X_2}(\tau)$ over all rules that depend only on the values of the Y_1's.

Now if Q_i is the absolute rank among all items of the item that is ith best of the n relatively best items, we have

$$X_2(k) = Q_{X_1(k)}.$$

This problem, therefore, reduces to a finite secretary problem with utility function $U_1(k) = E[U_{Q_k}]$, since the properties of the random variables $Q_1, \ldots, Q_n$ enable one to prove

$$E[U_{x_2(\tau)}] = E[U_1(X_1(\tau))].$$

The minimal expected cost does not exceed those for the standard finite problem and for the full memory discretized problem.

8 Miscellaneous variations

We collect together in this section papers which do not fit into any of the preceding categories.

8.1 Limited recall, minimum expected rank

Goldys (1978) allows the observer to accept the most recently presented item or the one before it. Govindarajulu (1975) had previously shown that the optimal policy says stop at the rth item if either

$$s_r \leqslant \min \left((r+1)c_r/(n+1), s_{r-1} \right)$$

or

$$s_{r-1} \leqslant \min \left((r+1)c_r/(n+1), s_r - 1 \right),$$

where s_{r-1}, s_r denote the apparent ranks of the $(r-1)$th and rth items. He gave the necessary recurrence relations for the $\{c_r\}$. Goldys extends the method of Chow *et al.* (1964) to show that as $n \to \infty$ the minimum expected rank tends to

$$\prod_{j=1}^{\infty} \left(\frac{j+1}{j} \right)^{2/(2j+1)} \simeq 2.57.$$

8.2 Play against an opponent

Gilbert & Mosteller (1966) considered the standard problem in which the observer plays against an opponent who is allowed to choose the order in which the items are presented so as to try to minimize the observer's probability of winning. If the opponent has a completely free choice of order, he should choose at random any line from the cyclic $n \times n$ Latin square, since this reduces the probability of winning to the minimum possible value of $1/n$ whatever strategy the observer uses. Suppose, however, the opponent is only allowed to choose the position of the best item in the order, the remaining items being equally likely to be in any one of the $(n-1)!$ possible orders. If the best item is placed in position r, call this strategy T_r and denote by T the randomized strategy that chooses T_r with probability p_r.

Suppose further that the observer can only use strategies like S_i 'ignore the first i items, choose the first relatively best item thereafter'. Denote by S the strategy that chooses S_i with probability π_i. Now if the observer uses strategy S_i and the opponent uses T_r, the probability of winning is 0 if $i \geqslant r$ and $i/(r-1)$ if $i < r$, since this is the probability that the best item out of the first $r-1$ items comes in the first i. The probability of winning using strategy S_i is therefore

$$\sum_{r=i+1}^{n} p_r \frac{i}{r-1}.$$

The opponent will naturally choose his $\{p_r\}$ so as to make this the same for all i, giving $p_r = K/r$ and $p_n = K$, where

$$K = \left\{ 1 + \sum_{j=1}^{n-1} \frac{1}{j} \right\}^{-1}.$$

The probability of winning is then K.

Similarly, if the opponent uses T_r and the observer uses S his probability of winning is

$$\sum_{i=0}^{r-1} \pi_i \frac{i}{r-1}.$$

He naturally chooses his $\{\pi_i\}$ to make this the same for all r, giving $\pi_0 = K$, $\pi_i = K/i$ $(i = 1, 2, \ldots, n-1)$ and the probability of winning is again K. This is therefore the true minimax solution of the two-person game. For large n the value of the game is asymptotically $\{1 + \gamma + \log(n-1)\}^{-1}$, where γ is Euler's constant. This therefore tends to 0 as $n \to \infty$ in constrast with the standard problem.

Gilbert & Mosteller repeat this analysis for the more complex case when the observer is allowed two choices. They once again find the minimax solution and show that the value of the game is

$$\left(2 - \frac{1}{n-1} \right) \left\{ 2 + \sum_{j=2}^{n-2} \frac{1}{j} \right\}^{-1}.$$

roughly double what it was before.

Chow et al. (1964) consider trying to minimize the expected rank of the accepted item when the order in which the items are presented is chosen by an opponent who is trying to maximize the rank of the item the observer accepts. By choosing a number r at random between 1 and n and then accepting the rth item, the observer can achieve an expected rank $n^{-1} \sum r = \frac{1}{2}(n+1)$, where the sum is over $r = 1, \ldots, n$, whatever order the opponent chooses. They show, remarkably, that the opponent has a strategy that can prevent the observer from doing any better than this however hard he tries. This consists of, for each item presented, choosing either the best or the worst item, each with probability $\frac{1}{2}$, out of those items that have not been presented so far. This is best illustrated by Fig. 1, where the numbers denote x_i, the true rank of the ith item presented.

If y_i denotes the apparent rank of the ith item and z_i is defined by

$$z_i = E(x_i \mid y_1 \ldots y_i),$$

it is easy to see that $\{z_i\}$ forms a martingale. Thus, for any stopping time τ, $E(z_\tau) = E(z_1) = \frac{1}{2}(n+1)$.

Irle & Schmitz (1979) allow general nonincreasing utilities as in §4 and discounting as
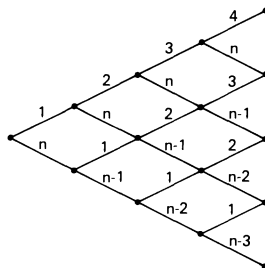

Figure 1. *Tree diagram showing true ranks of items presented by opponent using optimal strategy*

in § 2.2. The game of observer versus opponent again has a minimax solution with values $\sum_j U_j / \sum_j d^{-j}$, where the sums are over $j = 1, \ldots, n$. The observer's best strategy is to accept the rth item with probability $d^{-r} / \sum_j d^{-j}$ and the opponent presents items just as in Fig. 1, but not now with equal probabilities to each branch.

8.3 Finite memory

Rubin & Samuels (1977) consider problems in which the observer is only allowed to remember just one of the previously presented items. That is, the only thing he can observe about a current item is whether it is better or worse than the previously remembered one. One motivation is that, for the standard finite problem, the optimal policy is just such a finite memory one, since it is only necessary to remember the best item seen so far.

For each item, the observer now has three choices; to accept it and stop, to reject it, or to remember it and forget the previously remembered one. His optimal policy can be described by a sequence of choices $\{W_r/B_r; \ r = 2, 3, \ldots, n-1\}$, where W_r and B_r denote any of accept, reject or remember, W_r being the choice if the rth item is worse than the previously remembered one and B_r the choice if it is better. For the standard problem, for example, the optimal policy is

$$W_r/B_r = \begin{cases} \text{reject/remember} & (r \leqslant r^*), \\ \text{reject/accept} & (r > r^*). \end{cases}$$

For the finite problem minimizing the expected rank, Rubin & Samuels find the optimal finite memory policy among the class that only includes three out of the nine possible choices, adding remember/remember to the above two.

This is of the form

$$W_r/B_r = \begin{cases} \text{reject/remember} & (r < a_n), \\ \text{reject/accept} & (a_n \leqslant r < r_n), \\ \text{remember/remember} & (r = r_n), \\ W'_{r-r_n}/B'_{r-r_n} & (r > r_n), \end{cases}$$

where $\{W'_i/B'_i; \ i = 1, \ldots, n - r_n\}$ is the optimal policy for the same problem with $n - r_n + 1$ items. Recursive equations exist for finding a_n and r_n.

Turning to the infinite problem, the remarkable result is that the minimal expected rank of the accepted item remains finite even with the finite memory constraint. Consider only the class of policies that choose numbers

$$0 = R_0 < A_1 < R_1 < \ldots < A_k < R_k < \ldots < 1$$

and alternately remembers the best item in each (R_{k-1}, A_k) and then accepts the first item in (A_k, R_k) better than the remembered one. The minimal expected rank is 7.41 and the best values of the A's and R's are

$$R_{k+1} = R_k + R_1(1 - R_1)^k, \quad A_{k+1} = R_k + pR_1(1 - R_1)^k,$$

where $R_1 = 0.456$ and $p = 0.296$.

Even stronger, the expected loss remains finite when the loss function $q(k)$ increases as a power of k.

There remain many unsolved problems here. The nature of truly optimal policies for both finite and infinite problems is still unknown. The extension to problems that permit remembering m previously presented items is also unexplored.

8.4 Observation cost

Lorenzen (1978) first changed the utility function to allow the cost of each observation, a problem which he pursued further (Lorenzen, 1981). He allowed general nonincreasing utilities as in § 4 with finite $U = \lim U_i$ as $i \to \infty$, and defined $h_n(r)$ as the cost of observing the first r items out of n. Two possible assumptions about cost were explored:

(a) define a bounded nondecreasing sequence $\{h(i)\}$ for $i = 1, 2, \ldots$, and for finite n let $h_n(i) = h(i)$ $(i = 1, \ldots, n)$;

(b) define an increasing function $h(.)$ on $[0, 1]$ and for finite n let $h_n(i) = h(i/n)$.

Thus, for (a), increasing n merely adds extra numbers to the $h_n(i)$ set, while for (b) it decreases all previous $h_n(i)$ as well.

The solution to (a) turns out to be trivial. It is optimal either to accept the first item, gaining $U - h(1)$ or to use the optimal policy without observation costs.

For (b) a Mucci-style analysis leads as $n \to \infty$ to the differential equation

$$f'(x) = -\frac{1}{x} \sum_{s=1}^{\infty} [R_s(x) + h(x) - f(x)]^+, \quad f(1) = U - h(1),$$

and its corresponding solution. This equation also governs the appropriate modification of Gianini & Samuels' infinite problem, and the maximum expected utility again is $f(0)$. The optimal policy is, in general, an island rule, and it was shown that if

$$W_s(x) = \frac{1}{x} \sum_{j=1}^{s} [R_{s+1}(x) - R_j(x)] - h'(x)$$

has at most one sign change from $+$ to $-$ for each s, then there is a single island, returning us to a simple cut-off rule.

Bartoszynski & Govindarajulu (1978) obtain explicit results for a special case of the above with $U_1 = a$, $U_2 = b$, $U_3 = \ldots = 0$.

8.5 Sampling from an urn

Chen & Starr (1980) consider an urn containing balls labelled 1 to n which are sampled without replacement. If the observer stops after drawing the rth ball, he receives utility $f(r, m_r)$, where m_r denotes the largest number on any of the first r balls. Thus, not only is complete recall allowed, but the actual ranks of the balls are known as they are drawn. This puts the paper rather far from a recognizable secretary problem and it will not be described further here.

9 Full and partial information

As mentioned in § 1.2, the origins of the secretary problem lie with Cayley (1875), where values are observed sequentially from a known distribution. Gilbert & Mosteller call this the 'full information' problem since the observer knows at each stage as much as he can ever know about the next observation. In this sense the standard secretary problem can be called 'no information' as the observer is presented with values from a completely unknown distribution and he sees only their relative ranks and knows only that all rank orders are equally likely. An intermediate problem of 'partial information' occurs when observations are taken from a distribution of known form but containing one or more unknown parameters. A natural approach is to use Bayes theorem to update knowledge about these parameters at the same time as deciding whether to stop or continue.

Both these problems must be regarded as outside the scope of this review, so the reader is referred to the following papers for more details:

   (i) full information: Moser (1956), Guttman (1960), Karlin (1962), Gilbert & Mosteller (1966), Enns (1970), Sakaguchi (1973, 1976, 1978), Petrucelli (1982);

  (ii) partial information: Sakaguchi (1961) and DeGroot (1968) normal distribution, unknown mean; Campbell (1977) Dirichlet process; Bowerman & Koehler (1978) general distribution, one unknown parameter, sampling cost, complete recall; Petrucelli (1978) normal, exponential, inverse power location-scale families; Samuels (1978) and Stewart (1978) uniform, unknown endpoints.

References

Abdel-Hamid, A.R., Bather, J.A. & Trustrum, G.B. (1982). The secretary problem with an unknown number of candidates *J. Appl. Prob.* **19**, 619–630.

Bartoszynski, R. (1974). On certain combinatorial identities. *Colloquium Mathematicum* **30**, 289–293.

Bartoszynski, R. (1976). Some remarks on the secretary problem. *Commentationes Mathematicae Prace Matematycz ne* **19**, 15–22.

Bartoszynski, R. & Govindarajulu, Z. (1978). The secretary problem with interview cost. *Sankhyā* B **40**, 11–28.

Bissinger, B.H. & Siegel, C. (1963). Problem 5086. *Am. Math. Mon.* **70**, 336.

Bosch, A.J. (1964). Solution to problem 5086. *Am. Math. Mon.* **71**, 329–330.

Bowerman, B.L. & Koehler, A.B. (1978). An optimal policy for sampling from uncertain distributions. *Comm. Statist.* A **7**, 1041–1051.

Breiman, L. (1964). Stopping rule problems. In *Applied Combinatorial Mathematics*, Ed. by E.F. Beckenback, pp. 284–319. New York: Wiley.

Campbell, G. (1977). The maximum of a sequence with prior information. Purdue University, Department of Statistics, Mimeograph series No. 485.

Cayley, A. (1875). Mathematical questions and their solutions. *Educational Times* **22**, 18–19.

Chen, W.-C. & Starr, N. (1980). Optimal stopping in an urn. *Ann. Prob.* **8**, 451–464.

Chow, Y.S. & Robbins, H. (1963). On optimal stopping rules, *Z. Wahr.* **2**, 33–49.

Chow, Y.S., Moriguti, S., Robbins, H. & Samuels, S.M. (1964). Optimal selection based on relative rank (the "secretary problem"), *Israel J. Math.* **2**, 81–90.

Chow, Y.S., Robbins, H. & Siegmund, D. (1971). *Great Expectations: The Theory of Optimal Stopping.* Boston: Houghton Mifflin Co.

Cowan, R. & Zabczyk, J. (1978). An optimal selection problem associated with the Poisson process. *Theory Prob. Applic.* **23**, 584–592.

DeGroot, M.H. (1968). Some problems of optimal stopping. *J. R. Statist. Soc.* B **30**, 108–122.

Dynkin, E.B. (1963). The optimal choice of the stopping moment for a Markov process. *Dokl. Akad. Nauk. SSSR* **150**, 238–240.

Enns, von E.G. (1970). The optimum strategy for choosing the maximum of N independent random variables. *Unternehmensforschung* **14**, 89–96.

Frank, A.Q. & Samuels, S.M. (1980). On an optimal stopping problem of Gusein-Zade. *Stoch. Processes. Applic.* **10**, 299–311.

Gardner, M. (1960a). Mathematical games. *Scientific American* **202** (2), 152.

Gardner, M. (1960b). Mathematical games. *Scientific American* **202** (3), 178–179.

Gianini, J. (1976). The secretary problem with random number of individuals. Unpublished.

Gianini, J. (1977). The infinite secretary problem as the limit of the finite problem. *Ann. Prob.* **5**, 636–644.

Gianini, J. & Samuels, S.M. (1976). The infinite secretary problem. *Ann. Prob.* **4**, 418–432.

Gianini-Pettitt, J. (1979). Optimal selection based on relative ranks with a random number of individuals. *Adv. Appl. Prob.* **11**, 720–736.

Gilbert, J. & Mosteller, F. (1966). Recognizing the maximum of a sequence. *J. Am. Statist. Assoc.* **61**, 35–73.

Goldys, B. (1978). The secretary problem—the case with memory for one step. *Demonstratio Mathematica* **11**, 789–799.

Govindarajulu, Z. (1975). The secretary problem: optimal selection with interview cost. Technical report 82, University of Kentucky.

Gusein-Zade, S.M. (1966). The problem of choice and the optimal stopping rule for a sequence of independent trials. *Theory Prob. Applic.* **11**, 472–476.

Guttman, I. (1960). On a problem of L. Moser. *Can. Math. Bull.* **3**, 35–9.

Henke, M. (1970). *Sequentialle Auswahlprobleme bei Unsicherheit.* Meisenheim: Anton Hain Verlag.

Henke, M. (1973). Expectations and variances of stopping variables in sequential selection processes. *J. Appl. Prob.* **10**, 786–806.

Irle, A. (1980). On the best choice problem with random population size. *Z. für Operations Research* **24**, 177–190.

Irle, A. & Schmitz, N. (1979). Minimax strategies for discounted secretary problems. *Operat. Res. Verfahren.* **30,** 77–86.

Karlin, S. (1962). Stochastic models and optimal policy for selling an asset. Chapter 9 of *Studies in Applied Probability and Management Science*, Ed. by K. Arrow, S. Karlin and W. Scarf, pp. 148–158. Stanford University Press.

Lindley, D.V. (1961). Dynamic programming and decision theory. *Appl. Statist.* **10,** 39–52.

Lorenzen, T. J. (1978). Generalising the secretary problem. *Adv. Appl. Prob.* **11,** 384–396.

Lorenzen, T. J. (1981). Optimal stopping with sampling cost: the secretary problem. *Ann. Prob.* **9,** 167–172.

Moser, L. (1956). On a problem of Cayley. *Scripta Mathematica* **22,** 289–292.

Mucci, A.G. (1973a). Differential equations and optimal choice problems. *Ann. Statist.* **1,** 104–113.

Mucci, A.G. (1973b). On a class of secretary problems. *Ann. Prob.* **1,** 417–427.

Nikolaev, M.L. (1977). On a generalisation of the best choice problem. *Theory Prob. Applic.* **22,** 187–190.

Petrucelli, J.D. (1978). Some best choice problems with partial information. Unpublished thesis, Worcester Polytechnic Institute.

Petrucelli, J.D. (1981). Best-choice problems involving uncertainty of selection and recall of observations. *J. Appl. Prob.* **18,** 415–425.

Petrucelli, J.D. (1982). Full-information best-choice problems with recall of observations and uncertainty of selection depending on the observation. *Adv. Appl. Prob.* **14,** 340–358.

Presman, E.L. & Sonin, I.M. (1972). The best choice problem for a random number of objects. *Theory Prob. Applic.* **17,** 657–668.

Rasche, M. (1975). Allgermeine Stopprobleme. Technical report, Institut für Mathematische Statistik, Universität Münster.

Rasmussen, W.T. (1972). Optimal choosing problems. Ph.D. Thesis, Dept of Operations Research, Stanford, California.

Rasmussen, W.T. (1975). A generalised choice problem. *J. Optimization Theory Applic.* **15,** 311–325.

Rasmussen, W.T. & Pliska, S.R. (1976). Choosing the maximum from a sequence with a discount function. *Appl. Math. Optimization* **2,** 279–289.

Rasmussen, W.T. & Robbins, H. (1975). The candidate problem with unknown population size. *J. Appl. Prob.* **12,** 692–701.

Rubin, H. (1966). The "secretary" problem (Abstract). *Ann. Math. Statist.* **37,** 544.

Rubin, H. & Samuels, S.M. (1977). The finite memory secretary problem. *Ann. Prob.* **5,** 627–635.

Sakaguchi, M. (1961). Dynamic programming of some sequential sampling design. *J. Math. Anal. Applic.* **2,** 446–466.

Sakaguchi, M. (1973). A note on the dowry problem. Rep. Stat. Appl. Res., *JUSE* **20,** 11–17.

Sakaguchi, M. (1976). Optimal stopping problems for randomly arriving offers. *Math. Japonicae* **21,** 201–217.

Sakaguchi, M. (1978). Dowry problems and OLA policies. Rep. Stat. Appl. Res., *JUSE* **25,** 124–128.

Sakaguchi, M. (1979). A generalised secretary problem with uncertain employment. *Math. Japonica* **23,** 647–653.

Samuels, S.M. (1978). Minimax stopping rules when the distribution is uniform with unknown endpoints. Technical report 523, Department of Statistics, Purdue University.

Smith, M.H. (1975). A secretary problem with uncertain employment. *J. Appl. Prob.* **12,** 620–624.

Smith, M.H. & Deely, J.J. (1975). A secretary problem with finite memory. *J. Am. Statist. Assoc.* **70,** 357–361.

Stewart, T.J. (1978). Optimal selection from a random sequence with learning of the underlying distribution. *J. Am. Statist. Assoc.* **73,** 775–780.

Stewart, T.J. (1981). The secretary problem with an unknown number of options. *Oper. Res.* **29,** 130–145.

Tamaki, M. (1979a). Recognizing both the maximum and the second maximum of a sequence. *J. Appl. Prob.* **16,** 803–812.

Tamaki, M. (1979b). A secretary problem with double choices. *J. Oper. Res. Soc. Japan.* **22,** 257–264.

Yang, M.C.K. (1974). Recognising the maximum of a random sequence based on relative rank with backward solicitation. *J. Appl. Prob.* **11,** 504–512.

Résumé

Ce résumé trace le développement dès son origine pendant la première période des années 60 de ce qu'on appelle le problème du secrétaire. Il passe en revue tous les travaux sur ce problème et ses extensions qui ont déjà été publiés.

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